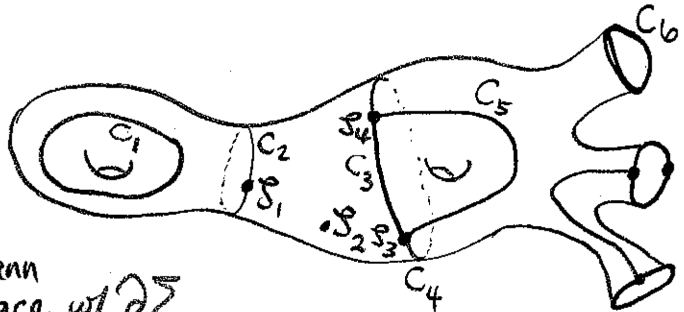
① Quilts and A_∞ structure.

Quilt = Σ
Riemann surface w/ $\partial\Sigma$

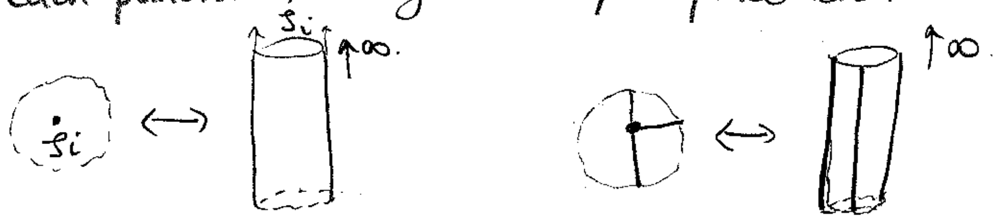


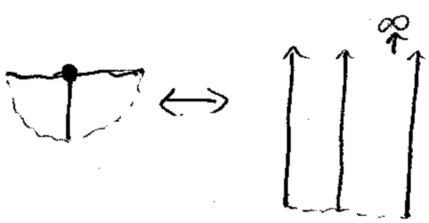
$\{C_i\}$ = collection of real analytic submanifolds "seams", & components of $\partial\Sigma \setminus \{\varsigma_i\}$ "boundary components"

$\{\varsigma_i\}$ = punctures

$\{\Sigma_i\}$ = components of $\Sigma \setminus \{C_i\}_s \,\&\, \{\varsigma_i\}_s$ "patches"

At each puncture, a cylindrical / striplike end.



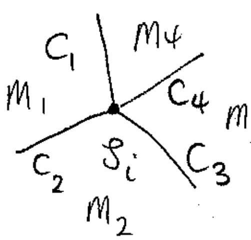


(Seams approach punctures as straight lines)

<u>Pseudohol. quilt</u>: Label $\Sigma_i \longleftrightarrow (M_i, \omega_i, J_i)$, $C_j \subset \partial\Sigma_i \cap \partial\Sigma_{i'}$

Label with $L_j \subset M_i^- \times M_{i+1}$
(Bdry comps labeled by Lagrangians)

Label ς_i with $a_i \in I(\underline{L})$ = gen. of CF (cyclic sequence of Lag. correspondences labeling seams incident to ς_i)



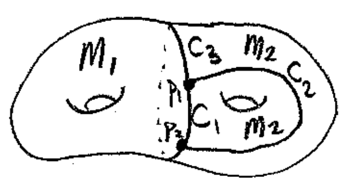
$M_3 \rightsquigarrow \mathrm{CF}(C_1, C_2, C_3, C_4)$.

Pseudohol. quilt: $\underline{u} = \{u_i\}$ $\quad u_i : \Sigma_i \longrightarrow M_i$ $\quad J_i$-hol. (<u>really</u>, w/ perturbations)

$C \subset \partial\Sigma_i \cap \partial\Sigma_{i'}$: $(u_i, u_{i'})\big|_C \subset L_{i,i'}$, finite energy, limits at ς_i's given by a_i's.

(2.)

E.g.



$p_1 \in \mathcal{I}(C_1, C_2, C_3)$

$p_2 \in \mathcal{I}(C_1, C_3, C_2)$

A_∞ structure from (genus zero!) quilts.

- **A_∞ bimodules.**

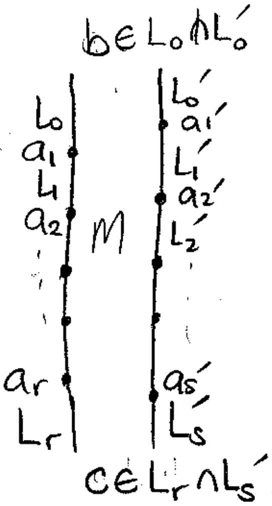
First ex: the diagonal bimodule.

$(\mathcal{F}(M), \mathcal{F}(M))$-bimodule

$L, L' \longmapsto CF(L, L')$

$\mu^{r|1|s}$:

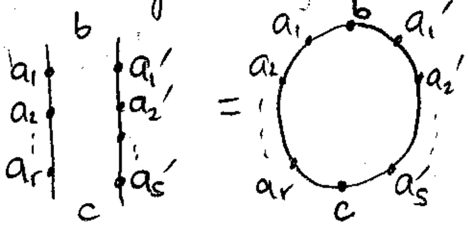
Picture:



Coeff. of c in

$\mu^{r|1|s}\left(\begin{matrix} a_1, \\ \vdots \\ a_r \end{matrix}\ b,\ \begin{matrix} a_1', \\ \vdots \\ a_s' \end{matrix}\right) = \mu^{r+s+1}(a_s', \dots, a_1', b, a_1, \dots, a_r)$

is count of isolated pseudohol. strips with markings as labeled.



$\mathcal{A}, \mathcal{A}'$ A_∞ categories.

$\mathcal{B}$ an $(\mathcal{A}, \mathcal{A}')$-bimodule means:

- $(X, Y) \in \mathrm{Ob}\,\mathcal{A} \times \mathrm{Ob}\,\mathcal{A}' \longmapsto \mathcal{B}(X, Y)$ chain complex.

- Bimodule operations: $r, s \geq 0$

$\mu^{r|1|s}$ $X_0, \dots, X_r$ $\mathrm{Ob}\,\mathcal{A}$

$Y_0, \dots, Y_s$ $\mathrm{Ob}\,\mathcal{A}'$

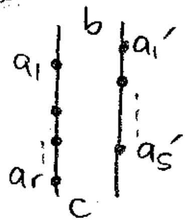
$$\mu^{r|1|s} : \begin{pmatrix} \hom_{\mathcal{A}}(X_0, X_1) \\ \otimes \\ \vdots \\ \otimes \\ \hom_{\mathcal{A}}(X_{r-1}, X_r) \end{pmatrix} \otimes \mathcal{B}(X_0, Y_0) \otimes \begin{pmatrix} \hom_{\mathcal{A}'}(Y_0, Y_1) \\ \otimes \\ \vdots \\ \otimes \\ \hom_{\mathcal{A}'}(Y_{s-1}, Y_s) \end{pmatrix} \longrightarrow \mathcal{B}(X_r, Y_s)$$

- multi-linear
- satisfy A_∞ bimodule equations:

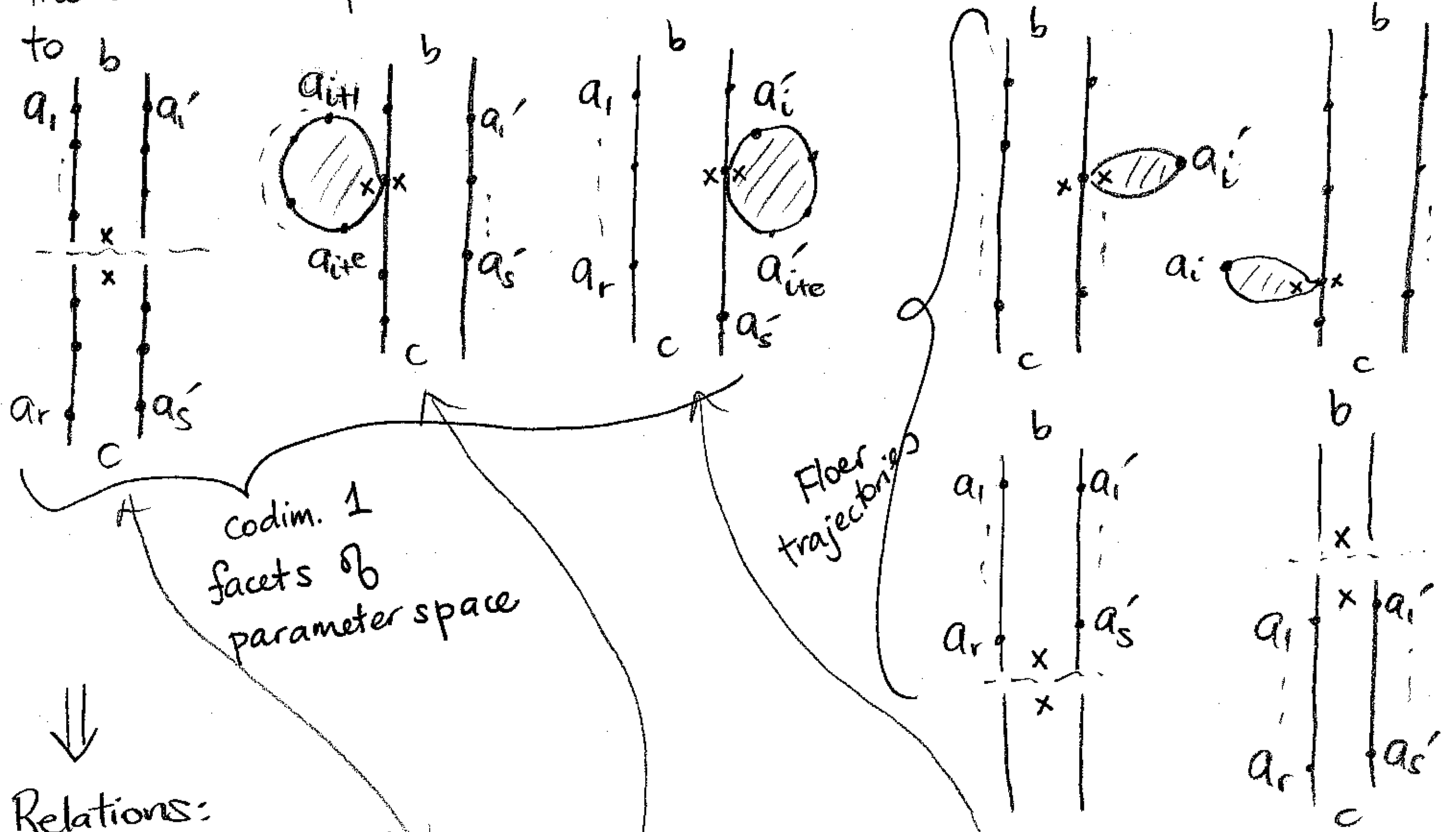
③

A_∞ bimodule relations:

1-dim^l moduli space of pseudohol. strips



<u>Compactify</u>: in absence of sphere/disk bubbling, the ends correspond to



$\Downarrow$

<u>Relations:</u>

$$0=\sum (\text{signs})\, \mu^{r-k|1|s-l}\left(\begin{pmatrix} a_{k+1}\\ \vdots \\ a_r\end{pmatrix}, \mu^{k|1|l}\left(\begin{pmatrix} a_1\\ \vdots \\ a_k\end{pmatrix}, b, \begin{pmatrix} a_1'\\ \vdots \\ a_l'\end{pmatrix}\right), \begin{pmatrix} a_{l+1}'\\ \vdots \\ a_s'\end{pmatrix}\right)$$

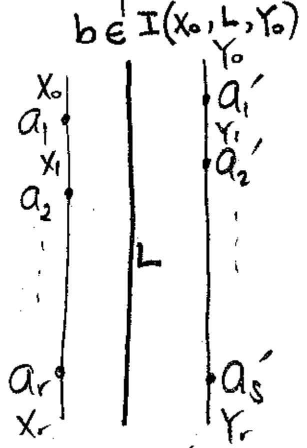
$$+\sum (\text{signs})\, \mu^{r-e+1|1|s}\left(\begin{pmatrix} a_1\\ \vdots \\ a_i \\ \mu^e(a_{i+1},\dots,a_{i+e}) \\ \vdots \\ a_r\end{pmatrix}, b, \begin{pmatrix} a_1'\\ \vdots \\ a_s'\end{pmatrix}\right) + \sum (\text{signs})\, \mu^{r|1|s-e+1}\left(\begin{pmatrix} a_1\\ \vdots \\ a_r\end{pmatrix}, b, \begin{pmatrix} a_1'\\ \vdots \\ a_i', \\ \mu^e(a_{i+1}',\dots, a_{i+e}') \\ \vdots \\ a_s'\end{pmatrix}\right)$$

④ Same structure if we allow seams.

$\rightsquigarrow$ $(\mathcal{F}(M), \mathcal{F}(N))$-bimodule associated to $L \subset M^- \times N$:

$$(\overset{\mathrm{ob}}{\underset{\uparrow}{X}}, \overset{\mathrm{ob}}{\underset{\uparrow}{Y}}) \longmapsto CF(X, L, Y) =: \mathcal{B}_L(X, Y).$$

Bimodule operations: $\mu^{r|1|s}$ counts isolated pseudohol. quilted strips



- In absence of sphere/disk bubbling, A_∞ bimodule relations hold.

Summarized as

Theorem: There is an A_∞ functor $\mathcal{G}$: $\mathrm{Fuk}(M^- \times N) \longrightarrow \mathrm{Bimod}(\mathrm{Fuk}(M), \mathrm{Fuk}(N))$

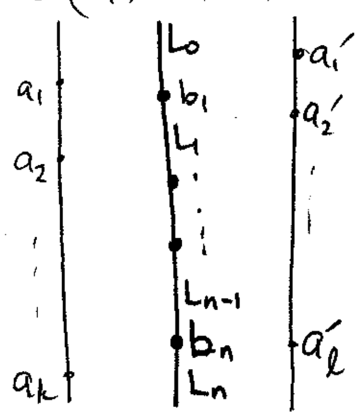
Proof / construction:

On objects: $L \subset M^- \times N \longmapsto \mathcal{B}_L = CF(\cdot, L, \cdot)$

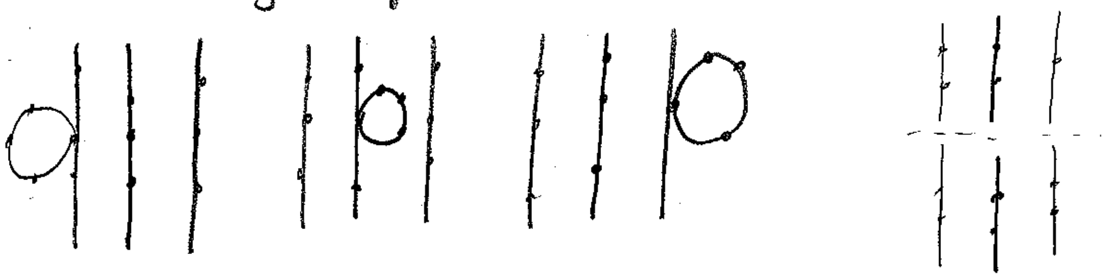
Morphisms: $\Phi^n : \hom(L_0, L_1) \otimes \cdots \otimes \hom(L_{n-1}, L_n) \longrightarrow \hom_{\mathrm{Bimod}}(\mathcal{B}_{L_0}, \mathcal{B}_{L_n})$.

$$b = (b_1, \ldots, b_n) \longmapsto \Theta_b = \{\Theta_b^{k|1|\ell}\}_{k \geq 0, \ell \geq 0}$$

Based on counts of



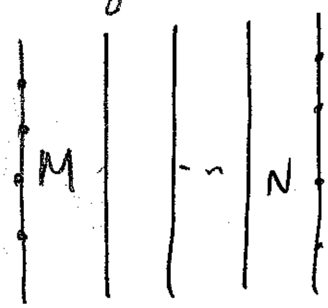
The fact it's an A_∞ functor follows directly from relations obtained by considering 1-dim$^{\underline{l}}$ moduli spaces (in absence of sphere/disk bubbling, always),



+ Floer trajectories.

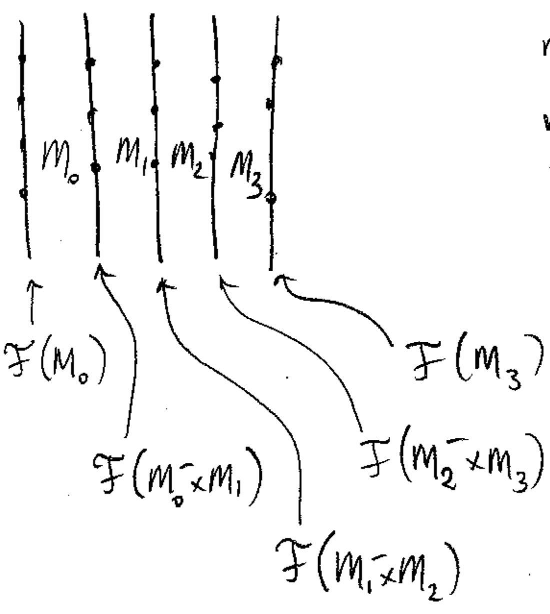
- More generally, any seq. of Lag. correspondences from M to N gives rise to a Bimodule,

via



- Quilted Floer chain groups really have structure of A_∞ "n-module",

e.g.



$n=1 \Rightarrow A_\infty$ module

$n=2 \Rightarrow$ bimodule.